# Improved Visibility Representation of Plane Graphs*

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## Abstract

In a visibility representation (VR for short) of a plane graph $G$, each vertex of $G$ is represented by a horizontal line segment such that the line segments representing any two adjacent vertices of $G$ are joined by a vertical line segment. Rosenstiehl and Tarjan [6], Tamassia and Tollis [9] independently gave linear time VR algorithms for 2-connected plane graph. Using this approach, the height of the VR is bounded by $(n-1)$, the width is bounded by $(2n-5)$. After that, some work have been done to find a more compact VR. Kant and He [3] proved that a 4-connected plane graph has a VR with width bounded by $(n-1)$. Kant [2] reduced the width bound to $\lfloor \frac{3n-6}{2} \rfloor$ for general plane graphs. Recently, using a sophisticated greedy algorithm, Lin et. al. reduced the width bound to $\lfloor \frac{22n-42}{15} \rfloor$ [5].

In this paper, we prove that any plane graph $G$ has a VR with width at most $\lfloor \frac{13n-24}{9} \rfloor$, which can be constructed by using the simple standard VR algorithm in [6, 9].

Keywords: Graph; Visibility representation

## 1  Introduction

A *visibility representation* (VR for short) of a plane graph $G$ is a drawing of $G$, where the vertices of $G$ are represented by non-overlapping horizontal segments (called *vertex segments*), and each edge of $G$ is represented by a vertical line segment touching the vertex segments of its end vertices. The problem of computing a compact VR is important not only in algorithmic graph theory, but also in practical applications such as VLSI layout [6, 9].

Without loss of generality, we assume that $G$ is a 2-connected plane graph with $n$ vertices. It was shown in [6, 9] that a VR of a plane graph $G$ can be obtained by a simple linear time algorithm from an *st-orientation* of $G$ and the corresponding *st*-orientation of its dual graph $G^*$. Using this approach, the height of the VR is bounded by $(n-1)$ and the width of the VR is bounded by $(2n-5)$ [6, 9].

One of the main concerns for VR is the size of the representation. Some work have been done to reduce the width of the VR by carefully choosing a special *st*-orientation of $G$. Kant and He proved that every 4-connected plane graph $G$ has a VR with width bounded by $(n-1)$ [3]. Based on this, Kant proved that every plane graph has a VR with width at most $\lfloor \frac{3n-6}{2} \rfloor$ [2]. Very recently, using a more sophisticated greedy algorithm, Lin et. al. reduced the width bound to $\lfloor \frac{22n-42}{15} \rfloor$ by choosing the best *st*-orientation from three *st*-orientations derived from a *Schnyder's realizer* of $G$ [5]. Their algorithm runs in linear time and uses a sophisticated greedy approach.

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In this paper, we prove that every plane graph $G$ has a VR with width at most $\lfloor \frac{13n-24}{9} \rfloor$ by applying the simple standard VR algorithm in [6, 9]. In order to obtain the new width bound, we first prove that, for a plane graph $G$ with an $st$-orientation $\mathcal{O}$, the width of the VR constructed by the algorithm in [6, 9] is bounded by $|E| - Score_{\mathcal{O}}(G)$, where $Score_{\mathcal{O}}(G)$ is the sum of the minimum of the number of bigger neighbors and smaller neighbors of vertices in $G$. Then we prove that, among the three $st$-orientations from a Schnyder's realizer of a plane triangulation $G$, at least one $st$-orientation $\mathcal{O}$ has $Score_{\mathcal{O}}(G) \geq \lceil \frac{14n-30}{9} \rceil$. Thus we are able to obtain the new width bound $\lfloor \frac{13n-24}{9} \rfloor$.

The present paper is organized as follows. Section 2 introduces definitions and preliminary results. Section 3 proves the relation between the width of VR and the score of an $st$-orientation for a 2-connected plane graph. Section 4 presents the construction of a VR with width at most $\lfloor \frac{13n-24}{9} \rfloor$.

## 2    Preliminaries

In this section, we give definitions and preliminary results. $G = (V, E)$ denotes a graph with $n = |V|$ vertices and $m = |E|$ edges. The *degree* of a vertex $v \in V$, denoted by $deg_G(v)$, is the number of edges incident to $v$. If $G$ is clearly understood, we simply write $deg(v)$ for $deg_G(v)$. A *planar graph* $G$ is a graph which can be embedded on the plane without edge crossings. A *plane graph* is a planar graph with a fixed embedding. The embedding of a plane graph divides the plane into a number of regions, called *faces*. The unbounded region is the *exterior face*. Other regions are *interior faces*. The vertices and the edges on the exterior face are called *exterior vertices* and *exterior edges*. Other vertices and edges are called *interior vertices* and *interior edges*. A *path* $P$ of $G$ is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ such that $(u_i, u_{i+1}) \in E$ for $1 \leq i < k$. We also use $P$ to denote the set of the edges in it. Each $u_i$ for $1 < i < k$ is called *an internal vertex* of $P$. Furthermore, if $(u_k, u_1) \in E$, then $u_1, u_2, \ldots, u_k$ is called a *cycle*. We normally use $C$ to denote a cycle and the set of the edges of it. If $C$ has 3 vertices, it is called a *triangle*. A cycle $C$ of $G$ divides the plane into its interior region and exterior region. If all facial cycles of $G$ are triangles, $G$ is called a *plane triangulation*. We abbreviate the words "counterclockwise" and "clockwise" as ccw and cw respectively.

The dual graph $G^* = (V^*, E^*)$ of a plane graph $G$ is defined as follows: For each face $F$ of $G$, $G^*$ has a node $v_F$. For each edge $e$ in $G$, $G^*$ has an edge $e^* = (v_{F_1}, v_{F_2})$ where $F_1$ and $F_2$ are the two faces of $G$ with $e$ on their common boundaries. $e^*$ is called the *dual edge* of $e$. For each vertex $v \in V$, the dual face of $v$ in $G^*$ is denoted by $v^*$.

An *orientation* of a graph $G$ is a digraph obtained from $G$ by assigning a direction to each edge of $G$. We will use $G$ to denote both the resulting digraph and the underlying undirected graph unless otherwise specified. (Its meaning will be clear from the context).
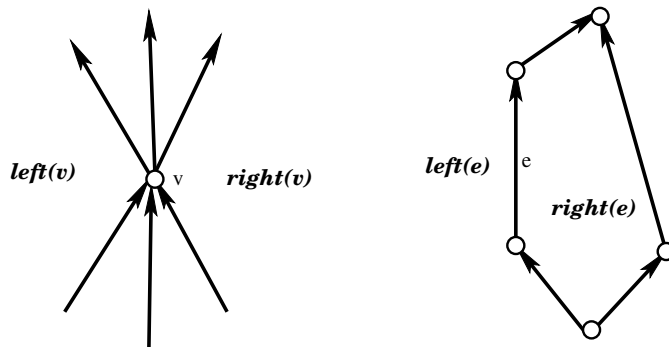


Figure 1: Properties of an $st$-orientation.

Let $G$ be a plane graph with a *distinguished* exterior edge $(s,t)$. An orientation of $G$ is called an *st-orientation*, if it is acyclic with $s$ as the only source and $t$ as the only sink.

Let $G$ be a plane graph and $(s,t)$ a distinguished exterior edge of $G$. An *st-numbering* of $G$ is a one-to-one mapping $\xi : V \to \{1, 2, \cdots, n\}$, such that $\xi(s) = 1$, $\xi(t) = n$, and each vertex $v \neq s, t$ has at least two neighbors $u, w$ with $\xi(u) < \xi(v) < \xi(w)$, where $u$ ($w$, resp.) is called a *smaller (bigger, resp.) neighbor* of $v$. Given an *st-numbering* $\xi$ of $G$, we can orient $G$ by directing each edge in $E$ from its lower numbered end vertex to its higher numbered end vertex. The resulting orientation will be called the *orientation derived from* $\xi$. Obviously, this orientation is an *st-orientation* of $G$. On the other hand, if $G = (V, E)$ has an *st-orientation* $\mathcal{O}$, we can define a 1-1 mapping $\xi : V \to \{1, \cdots, n\}$ by using topological sort. It is easy to see that $\xi$ is an *st-numbering* (where $\xi(s) = 1$ and $\xi(t) = n$) and the orientation derived from $\xi$ is $\mathcal{O}$. From now on, we will interchangeably use the term an *st-numbering* of $G$ and the term an *st-orientation* of $G$, where each edge of $G$ is directed accordingly.

Given any *st-orientation* of $G$, for each vertex $v$, the incoming edges of $v$ appear consecutively around $v$, and so do the outgoing edges of $v$. The face of $G$ that separates the incoming edges of $v$ from the outgoing edges of $v$ in clockwise direction is denoted by *left(v)*. The face of $G$ that separates the incoming edges of $v$ from the outgoing edges of $v$ is denoted by *right(v)*. The boundary of every face of $G$ consists of two directed paths. For each edge $e$ of $G$, the face on the left (right, resp.) side of $e$ is denoted by *left(e)* (*right(e)*, resp.) (See Fig. 1.)

Lempel et. al. [4] showed that for every 2-connected plane graph $G$ and any exterior edge $(s,t)$, there exists an *st-numbering* $\xi$ of $G$. Thus, $G$ has an *st-orientation* derived from $\xi$, with $s$ as the only source and $t$ as the only sink.
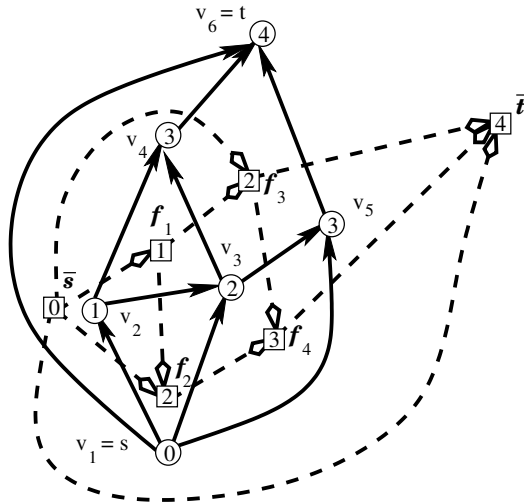


Figure 2: A 2-connected plane graph and one of its *st*-orientation $\mathcal{O}$.

Let $G$ be a 2-connected plane graph and $(s,t)$ be an exterior edge. Let $\mathcal{O}$ be an *st*-orientation of $G$, where $s$ is the source and $t$ is the sink. Consider the dual graph $G^*$ of $G$. For each $e \in G$, we direct its dual edge $e^*$ from the face on the left of $e$ to the face on the right of $e$ when we walk on $e$ along its direction in $\mathcal{O}$. We then reverse the direction of $(s,t)^*$. It was shown in [6, 9] that this orientation is an *st*-orientation of $G^*$ with $(s,t)^*$ as the distinguished exterior edge. We denote the source by $\bar{s}$, and the sink by $\bar{t}$. When we embed $G$ and $G^*$ on plane simultaneously, we fix $t^*$ to be the exterior face of $G^*$. We will denote this orientation of $G^*$ by $\mathcal{O}^*$ and call it the corresponding *st*-orientation of $\mathcal{O}$. For each vertex $v$ of $G$, we define $dist(v)$ to be the length of the longest path from source $s$ to $v$. For each node $v^*$ in $G^*$, define $dist^*(v^*)$ to be the length of the longest directed path from the source $\bar{s}$ to $v^*$.

For example, in Fig. 2, the graph $G$ drawn in solid lines is 2-connected with one $st$-orientation $\mathcal{O}$, $G^*$ is drawn in dashed lines with $\mathcal{O}^*$. The vertices of $G$ are represented by circles. The nodes of $G^*$ are represented by squares. The numbers inside the circles (squares, resp.) are the $dist(v)$ of vertices in $G$ ($dist^*(v^*)$ of nodes in $G^*$, resp.)

The following VR algorithm was given in [6, 9]:

**Algorithm 1: Visibility Representation**
**Input:** A 2-connected plane graph $G$.
**Output:** A VR of $G$.

1. Compute an $st$-orientation $\mathcal{O}$ of $G$.

2. Construct its dual $G^*$ and the corresponding $st$-orientation $\mathcal{O}^*$ of $G^*$.

3. Compute $dist(v)$ for the vertices of $G$ and $dist^*(v^*)$ for the nodes of $G^*$.

4. For each vertex $v$ of $G$, do:

   If $v \neq s, t$, draw horizontal line between $(dist^*(left(v)), dist(v))$ and $(dist^*(right(v)) - 1, dist(v))$.

   If $v = s$ or $t$, draw horizontal line between $(0, d(v))$ and $(dist^*(\bar{t}), d(v))$.

5. For each edge $(u, v)$ of $G$, draw vertical line between $(dist^*(left(u, v)), dist(u))$ and $(dist^*(left(u, v)), dist(v))$.

The correctness of the above VR drawing algorithm was proved in [6, 9]:

**Lemma 1** *Let $G$ be a 2-connected plane graph with $n$ vertices. Let $\mathcal{O}$ be an $st$-orientation of $G$. Algorithm 1 produces a VR of $G$ in linear time. The width of the VR is the length of the longest directed path in the corresponding $st$-orientation $\mathcal{O}^*$ of $G^*$.*

# 3   Bounding the Width of VR of a Plane Graph

We assume that $G$ is a 2-connected plane graph in this section. In order to obtain a VR of $G$ with reduced width, we need to find an $st$-orientation $\mathcal{O}$ of $G$ such that the length of longest path in $\mathcal{O}^*$ of $G^*$ is reduced. First, we introduce several concepts.

**Definition 1** *Let $G$ be a 2-connected plane graph, $\mathcal{O}$ be an $st$-orientation of $G$. Let $G^*$ be the dual graph of $G$, $\mathcal{O}^*$ be the corresponding $st$-orientation of $\mathcal{O}$.*

1. *For any vertex $v \neq s, t$ of $G$, define:*

   $Hand(v) = \{(v, u) \in E \mid u \text{ is a bigger neighbor of } v\}$;

   $Foot(v) = \{(u, v) \in E \mid u \text{ is a smaller neighbor of } v\}$.

   *For $v = s$, define $Foot(s) = \{(s, t)\}$, $Hand(s) = \{(s, u) \in E \mid u \neq t\}$. (Note, the two definitions for the source $s$ are special.)*

2. *For face $v^* \neq t^*$ of $G^*$, define:*

   $cover(v^*) = \{e^* \mid e \in Hand(v)\}$;

   $sheet(v^*) = \{e^* \mid e \in Foot(v)\}$.

   *Note that, for any face $v^* \neq t^*$ of $G^*$ (including $s^*$), its boundary consists of two directed paths, one is $cover(v^*)$, the other is $sheet(v^*)$.*

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3. For vertex $v \neq t$ of $G$, define $score_{\mathcal{O}}(v) = \min\{|Hand(v)|, \ |Foot(v)|\}$, and $Score_{\mathcal{O}}(G) = \sum_{v \neq t} score_{\mathcal{O}}(v)$.

For example, in Fig. 2, $Hand(v_3) = \{(v_3, v_4), \ (v_3, v_5)\}$ and $Foot(v_3) = \{(s, v_3), \ (v_2, v_3)\}$. For $v_3^*$ in $G^*$, $cover(v_3^*) = \{(f_1, f_3), \ (f_3, f_4)\}$ and $sheet(v^*) = \{(f_1, f_2), \ (f_2, f_4)\}$.

Now we can give the following technical theorem:

**Theorem 1** *Let $G$ be a 2-connected plane graph with an st-orientation $\mathcal{O}$. Let $\mathcal{O}^*$ be the st-orientation of $G^*$ derived from $\mathcal{O}$. Then $G$ has a VR with width at most $|E| - Score_{\mathcal{O}}(G)$.*

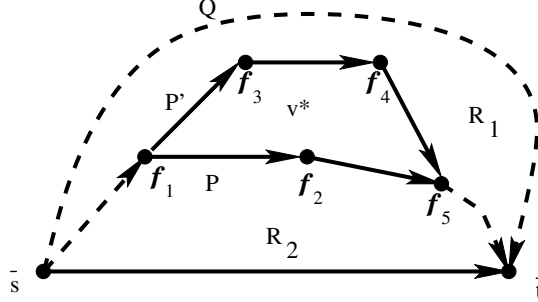**Proof:** We embed $G$ and $G^*$ in the plane, such that the face $t^*$ of $G^*$ as the exterior face of $G^*$. Let



Figure 3: The proof of Theorem 1.

$(\overline{s}, \overline{t})$ be the distinguished exterior edge of $\mathcal{O}^*$, $Q$ be the other path connecting $\overline{s}, \overline{t}$ which is part of the exterior boundary. Then $Q$ and $(\overline{s}, \overline{t})$ enclose a closed region, denoted by $R$, consisting of all the interior faces of $G^*$ (namely, all the dual faces $v^* \neq t^*$ of $v \neq t$ in $G$). See Fig. 3 for an illustration.

Let $P$ be the longest directed path in $\mathcal{O}^*$ from $\overline{s}$ to $\overline{t}$. $P$ cut the region $R$ into two subregions: $R_1$, $R_2$, where $R_1$ is enclosed by $Q$ and $P$, $R_2$ is enclosed by $P$ and $(\overline{s}, \overline{t})$. We observe that for any face $v^*$ inside $R_1$ ($R_2$, resp.), its $cover(v^*)$ ($sheet(v^*)$, resp.) is not on the path $P$. For example, in Fig. 3, $v^*$ is in $R_1$, $cover(v^*) = \{(f_1, f_3), \ (f_3, f_4), \ (f_4, f_5)\}$ is not on $P$. Also observe that: each edge in $G^*$ can only be in at most one cover set. Thus: $P \cap (\cup_{\{v^* \ in \ R_1\}} cover(v^*)) = \emptyset$. Similarly, $P \cap (\cup_{\{v^* \ in \ R_2\}} sheet(v^*)) = \emptyset$.

Note that all the interior faces of $G^*$ are either in $R_1$, or in $R_2$. So, $\sum_{\{v^* \ in \ R_1\}} |cover(v^*)| + \sum_{\{v^* \ in \ R_2\}} |sheet(v^*)| = \sum_{\{v| \ v^* \ in \ R_1\}} |Hand(v)| + \sum_{\{v| \ v^* \ in \ R_2\}} |Foot(v)| \geq Score_{\mathcal{O}}(G)$. Thus, the length of $P$ is at most $|E| - Score_{\mathcal{O}}(G)$. By Lemma 1, $G$ has a VR with width at most $|E| - Score_{\mathcal{O}}(G)$.
$\square$

In [5], Lin et. al. proved that their sophisticated greedy algorithm outputs a VR of $G$ with the width bounded by $|E| - Score_{\mathcal{O}}(G)$. Theorem 1 shows that the simpler VR algorithm in [6, 9] achieves the same width bound.

For a 4-connected plane triangulation $G$, Kant and He gave an st-orientation $\mathcal{O}$ of $G$ such that $Score_{\mathcal{O}}(G) \geq (2n - 5)$ [3]. Thus, using this st-orientation, Algorithm 1 outputs a VR of $G$ with width at most $(n - 1)$.

# 4 Compact Visibility Representation

Thus, in order to shorten the width of a VR of a plane graph $G$, we need to find an st-orientation $\mathcal{O}$ of $G$ such that $Score_{\mathcal{O}}(G)$ is as large as possible.

Without loss of generality, we assume that $G$ is a plane triangulation with $n \geq 4$ vertices in this section. First we need to introduce the concept of Schnyder's realizer [7, 8]:
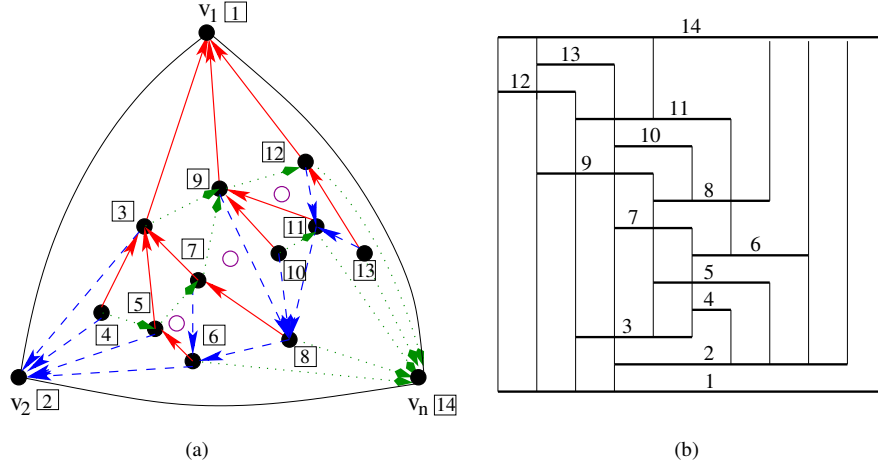
Figure 4: (a) A plane triangulation $G$ and one realizer $\mathcal{R}$ of $G$; (b) A VR of $G$.

**Definition 2** *Let $G$ be a plane triangulation with three exterior vertices $v_1, v_2, v_n$ in ccw order. A realizer $\mathcal{R}$ of $G$ is a partition of the interior edges of $G$ into three sets $T_1, T_2, T_n$ of directed edges such that the following hold:*

- *For each $i \in \{1, 2, n\}$, the interior edges incident to $v_i$ are in $T_i$ and directed toward $v_i$.*

- *For each interior vertex $v$ of $G$, the neighbors of $v$ form six blocks $U_1, D_n, U_2, D_1, U_n$, and $D_2$ in ccw order around $v$, where $U_j$ and $D_j$ ($j = 1, 2, n$) are the parent and the children of $v$ in $T_j$.*

It was shown in [7, 8] that every plane triangulation $G$ has a realizer $\mathcal{R}$, which can be obtained in linear time. Each $T_i$ ($i \in \{1, 2, n\}$) is a tree rooted at the vertex $v_i$ containing all interior vertices of $G$. Fig. 4 (a) shows a realizer of a plane triangulation $G$. Three trees $T_1$, $T_2$, $T_n$ are drawn as solid, dashed, and dotted lines, respectively. (Ignore the small boxes containing integers for now. Their meaning will be explained later.)

The following lemma shows how to obtain $st$-numberings from a Schnyder's realizer [10].

**Lemma 2** *Let $G$ be a plane triangulation and $\mathcal{R} = \{T_1, T_2, T_n\}$ a Schnyder's realizer of $G$. $T_i'$ be the tree obtained by $T_i$ plus the two exterior edges adjacent to $v_i$ in $G$. $T_i'$ is rooted at $v_i$. Then the ccw preordering of the vertices of $G$ with respect to $T_i'$ is an st-numbering of $G$.*

For example, consider the tree $T_1$ (rooted at $v_1$) shown in Fig. 4. The union of $T_1$ and the two exterior edges $(v_2, v_1)$ and $(v_n, v_1)$ is a tree of $G$, denote it by $T_1'$. The ccw preordering of the vertices of $G$ with respect to $T_1'$ are shown in integers inside the small boxes. It is an $st$-orientation of $G$ by Lemma 2, denoted by $\mathcal{O}_1$. Similarly, we have two other $st$-orientations $\mathcal{O}_2$, $\mathcal{O}_n$.

Denote the set of interior vertices of $G$ by $I$. Then for each vertex $v \in I$, $score_{\mathcal{O}_i}(v), i = 1, 2, n$ is always definable. And obviously $Score_{\mathcal{O}_i}(G) = 2 + \sum_{v \in I} score_{\mathcal{O}_i}(v)$. We denote $score_{sum}(v) = \sum_{i=1,2,n} score_{\mathcal{O}_i}(v)$ for each $v \in I$.

Next, we want to find a lower bound of $\sum_{i=1,2,n} Score_{\mathcal{O}_i}(G)$.

Let $inter(v) = \sum_{i=1,2,n}[v \ is \ not \ a \ leaf \ of \ T_i]$, where $[c]$ is 1 (0, resp.) if condition $c$ is true (false, resp.). Lin et. al. partitioned the interior vertices of $G$ into three subsets $A$, $B$, $C$ as follows [5]:

$A = \{v \mid inter(v) = 0\}$;
$B = \{v \mid inter(v) = 2, \ deg(v) = 5\}$;
$C = \{v \notin B \mid inter(v) \geq 1\}$.

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Let $\xi_i$ be the number of internal (namely, non-leaf) vertices in $T_i$. An interior face $f$ of $G$ is *cyclic* with respect to $R$ if each of its three edges belongs to different trees of $R$. Denote the number of cyclic interior faces with respect to $R$ by $\Delta(\mathcal{R})$. For example, in Fig. 4 (a), the faces $\{5,7,6\}$, $\{7,9,8\}$, $\{9,12,11\}$ (marked by empty circles) are the cyclic faces in $\mathcal{R}$. So, $\Delta(\mathcal{R}) = 3$.

The following results were proved in [1, 5]:

**Lemma 3** *Let $G$ be a plane triangulation with $n \geq 4$ vertices. Let $v_1, v_2, v_n$ be the exterior vertices of $G$ in the ccw order. Let $\mathcal{R} = \{T_1, T_2, T_n\}$ be any realizer of $G$, where $T_i$, $i \in \{1, 2, n\}$ is rooted at $v_i$ respectively. Let $k$ be the number of connected components of the graph $G[B]$, which is a subgraph of $G$ induced by $B$.*

1. *$\xi_1 + \xi_2 + \xi_n - \Delta(\mathcal{R}) = n - 1$.*

2. *$\xi_1 + \xi_2 + \xi_n - 3 = \sum_{v \in I} inter(v) \geq 2|B| + |C|$.*

3. *$score_{sum}(v) \geq 3 + 2 \cdot inter(v) - [v \in B]$, $v \in I$.*

4. *$|B| - k \leq 2\Delta(\mathcal{R})$.*

Now we can prove the following theorem:

**Theorem 2** *Let $G$ be a plane triangulation with $n \geq 4$ vertices, $R = \{T_1, T_2, T_n\}$ be any Schnyder's realizer of $G$. Then $\sum_{i=1,2,n} Score_{\mathcal{O}_i}(G) \geq \frac{14n}{3} - 10$.*

**Proof:** Let $G[B]$ be the subgraph of $G$ induced by $B$. Suppose that $G[B]$ has $k$ connected components. Let $|B| - k = \delta \Delta(\mathcal{R})$, then we have $0 \leq \delta \leq 2$ by Lemma 3 (4). Let $B_t$, $t = 1, 2, \cdots, k$ be all the connected components of $G[B]$. Lin et. al. observed that [5]: any two distinct vertices of $A$ are not adjacent in $G$; and each vertex in $A$ is adjacent to at most one $B_t$. Thus, we know that the number of the connected components of $G[A \cup B]$ is at least $k$. Considering $G - (A \cup B)$, each interior face of $G - (A \cup B)$ contains at most one connected component of $G[B]$. We remove edges of $G - (A \cup B)$ until each interior face contains exactly one connected component of $G[B]$, denote this graph by $G'$. (If $G[B]$ is empty, $G'$ does not have interior face.) Let $F_i$ $(i = 3, 4, \cdots)$ be the set of interior faces of $G'$ with $i$ edges on its boundary. Thus, we have:

$$k = \sum_{i=3}^{\infty} |F_i| \tag{1}$$

For any face in $F_i$, $i \geq 4$, we can triangulate it into $i - 2$ faces, then by applying Euler's formula to the resulting graph, its number of interior faces is at most $2(|C| + 3) - 5 = 2|C| + 1$. (3 comes from the exterior vertices of $G$.) Thus:

$$\sum_{i=3}^{\infty} (i - 2)|F_i| \leq 2(|C| + 3) - 5 = 2|C| + 1.$$

Therefore:

$$\sum_{i=3}^{\infty} \frac{i - 2}{2} |F_i| - \frac{1}{2} \leq |C| \tag{2}$$

Using Equation (1) and (2), we have:

$$|C| \geq \frac{k}{2} + \frac{1}{2}|F_4| + |F_5| + \frac{1}{2}|F_6| - \frac{1}{2} \tag{3}$$

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Applying Lemma 3 (1) and (2) and above equation, we have:

$$n + \Delta(\mathcal{R}) - 4 \geq 2|B| + |C| \;\geq\; 2k + 2\delta\Delta(\mathcal{R}) + \frac{k}{2} + \frac{1}{2}|F_4| + |F_5| + \frac{1}{2}|F_6| - \frac{1}{2}$$

$$= \; \frac{5}{2}k + 2\delta\Delta(\mathcal{R}) - \frac{1}{2} + \frac{1}{2}|F_4| + |F_5| + \frac{1}{2}|F_6|.$$

Thus, we have:

$$\frac{5}{2}k \leq n + \Delta(\mathcal{R}) - 2\delta\Delta(\mathcal{R}) - \frac{7}{2} - \frac{1}{2}|F_4| - |F_5| - \frac{1}{2}|F_6| \tag{4}$$

Because any vertex in $B$ has degree 5 in $G$, and an interior face of $G'$ in $F_3$ contains at least 1 vertex from $B$ in $G$, so it contains at least 3 vertices from $A \cup B$ in $G$. Similarly, an interior face of $G'$ in $F_4$ contains at least 2 vertices from $A \cup B$ in $G$. An interior face of $G'$ in $F_i$ for $i \geq 5$ contains at least 1 vertices from $A \cup B$ in $G$. Thus:

$$3|F_3| + 2|F_4| + \sum_{i=5}^{\infty} |F_i| \leq |A| + |B|.$$

Add this to Equation (2), we have:

$$\frac{7}{2}|F_3| + 3|F_4| + \frac{5}{2}|F_5| + 3|F_6| + \sum_{i=7}^{\infty} \frac{i}{2}|F_i| \leq |A| + |B| + |C| + \frac{1}{2} = n - \frac{5}{2}.$$

Combining it with Equation (1), we have:

$$\frac{7}{2}k \;\leq\; \left(\frac{7}{2}|F_3| + 3|F_4| + \frac{5}{2}|F_5| + 3|F_6| + \sum_{i=7}^{\infty} \frac{i}{2}|F_i|\right) + \left(\frac{1}{2}|F_4| + |F_5| + \frac{1}{2}|F_6|\right)$$

$$\leq \; n - \frac{5}{2} + \frac{1}{2}|F_4| + |F_5| + \frac{1}{2}|F_6| \tag{5}$$

Add Equation (4) and (5), and divide both sides by 6. We have:

$$k \leq \frac{n}{3} + \frac{1}{6}\Delta(\mathcal{R}) - \frac{1}{3}\delta\Delta(\mathcal{R}) - 1 \tag{6}$$

Applying Lemma 3 (2), (3) and above equation, also note that $0 \leq \delta \leq 2$, we have:

$$\sum_{i=1,2,n} Score_{\mathcal{O}_i}(G) \;=\; 6 + \sum_{v \in I} score_{sum}(v)$$

$$\geq \; 6 + \sum_{v \in I} \{3 + 2 \cdot inter(v)\} - |B|$$

$$= \; 6 + 3(n-3) + 2(n + \Delta(\mathcal{R}) - 4) - |B|$$

$$= \; 5n + 2\Delta(\mathcal{R}) - 11 - |B|$$

$$= \; 5n + 2\Delta(\mathcal{R}) - 11 - (|B| - k) - k$$

$$\geq \; 5n + 2\Delta(\mathcal{R}) - 11 - \delta\Delta(\mathcal{R}) - \frac{1}{3}n - \frac{1}{6}\Delta(\mathcal{R}) + \frac{1}{3}\delta\Delta(\mathcal{R}) + 1$$

$$= \; \frac{14n}{3} + \left(\frac{11}{6} - \frac{2}{3}\delta\right)\Delta(\mathcal{R}) - 10 \geq \frac{14n}{3} - 10 \tag{7}$$

$\square$

8

**Theorem 3** *Let $G$ be a plane triangulation with $n \geq 4$ vertices, then $G$ has a VR with width at most $\lfloor \frac{13n-24}{9} \rfloor$.*

**Proof:** Applying Theorem 2, we have $\sum_{i=1,2,n} Score_{O_i}(G) \geq \frac{14n}{3} - 10$. Thus, one of $Score_{O_i}(G) \geq \lceil \frac{14n-30}{9} \rceil$. Applying Theorem 1 and Algorithm 1, the width of the VR is at most $3n - 6 - \lceil \frac{14n-30}{9} \rceil = \lfloor \frac{13n-24}{9} \rfloor$. $\square$

For example, Fig. 4 (b) gives a VR of $G$, using the *st*-numbering in Fig. 4 (a).

Whether the bound $\frac{13n}{9} - O(1)$ is worst case optimal remains open.

# References

[1] N. Bonichon, B. L. Saëc and M. De la Libération, Wagner's theorem on realizers, in: Proc. ICALP'02, Lecture Notes in Computer Science, Vol. 2380, (Springer, Berlin, 2002) 1043-1053.

[2] G. Kant, A more compact visibility representation, International Journal of Computational Geometry & Applications 7 (1997) 197-210.

[3] G. Kant and X. He, Regular edge labeling of 4-connected plane graphs and its applications in graph drawing problems, Theoretical Computer Science 172 (1997) 175-193.

[4] A. Lempel, S. Even and I. Cederbaum, An algorithm for planarity testing of graphs, in: Theory of Graphs (Proc. of an International Symposium), Rome, 1966, 215-232.

[5] C.-C. Lin, H.-I. Lu and I-F. Sun, Improved Compact Visibility Representation of Planar Graph via Schnyder's Realizer, in: Proc. STACS'03, Lecture Notes in Computer Science, Vol. 2607, (Springer, Berlin, 2003) 14-25.

[6] P. Rosenstiehl and R. E. Tarjan, Rectilinear planar layouts and bipolar orientations of planar graphs, Discrete Comput. Geom. 1 (1986) 343-353.

[7] W. Schnyder, Planar graphs and poset dimension, Order 5 (1989) 323-343.

[8] W. Schnyder, Embedding planar graphs on the grid, in: Proc. the First Annual ACM-SIAM Symposium on Discrete Algorithms, 1990 138-148.

[9] R. Tamassia and I.G.Tollis, An unified approach to visibility representations of planar graphs. Discrete & Computational Geometry 1 (1986) 321-341.

[10] H. Zhang and X. He, Compact Visibility Representation and Straight-Line Grid Embedding of Plane Graphs, in: Proc. WADS'03, Lecture Notes in Computer Science, Vol. 2748, (Springer, Berlin, 2003) 493-504.